

SMALL-RECOIL APPROXIMATION

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Abstract

In this review we discuss a technique to compute and to sum a class of Feynman diagrams, and some of its applications. These are diagrams containing one or more energetic particles that suffer very little recoil in their interactions. When recoil is completely neglected, a decomposition formula can be proven. This formula is a generalization of the well-known eikonal formula, to non-abelian interactions. It expresses the amplitude as a sum of products of the irreducible amplitudes, with each irreducible amplitude being the amplitude to emit one, or several mutually interacting, quasi-particles. For abelian interaction a quasi-particle is nothing but the original boson, so this decomposition formula reduces to the eikonal formula. In non-abelian situations each quasi-particle can be made up of many bosons, though always with a total quantum number identical to that of a single boson. This decomposition enables certain amplitudes of all orders to be summed up into an exponential form, and it allows subleading contributions of a certain kind, which is difficult to reach in the usual way, to be computed. For bosonic emissions from a heavy source with many constituents, a quasi-particle amplitude turns out to be an amplitude in which all bosons are emitted from the same constituent. For high-energy parton-parton scattering in the near-forward direction, the quasi-particle turns out to be the Reggeon, and this formalism

shows clearly why gluons reggeize but photons do not. The ability to compute subleading terms in this formalism allows the BFKL-Pomeron amplitude to be extrapolated to asymptotic energies, in a unitary way preserving the Froissart bound. We also consider recoil corrections for abelian interactions in order to accommodate the Landau-Pomeranchuk-Migdal effect.

I. INTRODUCTION

Emission and absorption of soft particles cause hardly any recoil to an energetic source. Surprisingly, this trivial fact gives rise to a great deal of simplification in quantum field theoretical calculations. This is the subject we wish to review in the article.

Such a source with an energy $p^0 = \sqrt{\vec{p}^2 + m^2}$ may be *relativistic* when $|\vec{p}|$ is large and m is small, or *non-relativistic* when $|\vec{p}|$ is small and m is large. For example, a high-energy quark is a relativistic source of gluons, and a heavy nucleus at rest is a non-relativistic source of soft pions.

The origin of this simplification is a *decomposition formula* [1] for the tree amplitude A_n of n identical bosons. This formula allows A_n to be decomposed into a sum of products of *irreducible amplitudes* I_m , where m runs from 1 to n , labeling the number of bosons in it. The number k of irreducible factors in each term again varies from 1 to n , but to preserve the total number of bosons emitted the condition $\sum_{i=1}^k m_i = n$ must be obeyed. An irreducible amplitude I_m differs from a Feynman tree amplitude A_m only in having the product of vertex factors $V_1 V_2 \cdots V_{m-1} V_m$ in A_m replaced by their nested commutator $[V_1, [V_2, [\cdots, [V_{m-1}, V_m] \cdots]]$ in I_m . The decomposition formula is combinatorial in nature and will be discussed more fully in Sec. 2. It is valid whatever V_i 's are, and whether the bosons are on-shell or off-shell. The latter makes it possible for the tree diagram in question to be a part of a much larger Feynman diagram, thus allowing the decomposition of tree amplitudes to be applied fruitfully to loop diagrams as well.

The vertex factors V_i are matrices causing a change in the spin and internal quantum

numbers of the energetic source after each emission (or absorption). They are *abelian* if the spin and quantum numbers remain the same after each emission. In that case they are diagonal and commute with one another, so all I_m vanish except $I_1 = A_1$. As a result, the n -boson amplitude A_n is factorized into a product of n single-boson amplitudes A_1 . Such a decomposition for abelian vertices have been known for a long time under the name of an *eikonal formula* [2]. It is often used to demonstrate the cancelation of infrared divergence in QED [3–5], and to establish the geometrical nature of a scattering amplitude at high energy [6,7]. It is well documented in text books so we shall not discuss it any further until Sec. 5, when its recoil correction is considered.

Let us examine more carefully what decomposition means in the non-abelian context. We shall concentrate on the predominant situation when the vertices V_i are generators of a Lie group. In that case the emitted bosons carry the quantum numbers of the adjoint representation. Since a nested commutator of generators is a generator, the m bosons emitted in I_m also carry a total quantum number in the adjoint representation. We may therefore think of these m bosons together to form a *quasi-particle*, with the same quantum number as an original boson. The irreducible amplitude I_m is then an amplitude for the emission of a single quasi-particle. For that reason we shall use the terms ‘irreducible amplitude’ and ‘quasi-particle amplitude’ interchangeably. The decomposition formula then says multi-quasi-particle amplitudes are always factorizable into products of single-quasi-particle amplitudes, much like the abelian case. Thus the decomposition formula is also referred to as the *factorization formula*. In this language the only difference between abelian and non-abelian vertices is that a quasi-particle in the abelian case is the original boson, whereas in the non-abelian case it may have a complicated structure consisting of many bosons. This difference is eventually responsible for the gluon to reggeize and not the photon, because the quasi-particle in high-energy scattering turns out to be just the Reggeon. For more details see Sec. 4.

The decomposition formula is able to simplify calculation in at least three ways. First, all the terms in A_n with $k > 1$ can be computed from I_m with $m < n$, which in turn can be

computed from A_m with $m < n$. The only new term needed to be computed at order n is the quasi-particle amplitude I_n , thus reducing the labor of computation. Secondly, factorization often allows A_n of all n to be summed up, typically into an exponential function of I_m . This is important in problems where phase is of paramount concern, as in the unitarization of total cross section in Sec. 4, and the coherent reduction of radiation in the Landau-Pomeranchuk-Migdal (LPM) effect to be discussed later and in Sec. 5. Thirdly, quite often all I_m have the same order of magnitude μ , irrespective of what m is. This will be the case for the examples discussed in Secs. 3 and 4. In that case terms with the product of k irreducible amplitudes are of order μ^k , and it is important to note also that these are also terms with quantum numbers made up of k adjoint objects. This correlation between the magnitude and the quantum number is useful as we shall illustrate in Sec. 3. In particular, when $\mu \ll 1$, this means that magnitude is determined by the quantum number. To see that let \mathcal{C}_k be the set of internal quantum numbers first appearing in the tensor product of k adjoint objects. For example, in the case when the internal quantum number is $SU(3)$ color, octet alone is in \mathcal{C}_1 , whereas singlets, decuplets, anti-decuplets, and 27-plets are in \mathcal{C}_2 . In this notation, amplitudes belonging to \mathcal{C}_k is of order μ^k when $\mu \ll 1$. Though small, such subdominant amplitudes can be extracted in this approach simply by multiplying k dominant amplitudes I_m together, each of order μ . This is not something that can be obtained in the usual way short of carrying out the very difficult task of calculating each Feynman diagram to an accuracy of μ^k . These subdominant contributions are needed in Sec. 4 to preserve the Froissart bound when total cross section is extrapolated to asymptotic energies.

These advantages of the decomposition formula will be illustrated with two concrete examples. In Sec. 3, we consider a source made up of $N \gg 1$ constituents. That source may be a heavy nucleus emitting soft gluons [8,9], or a baryon emitting soft pions in a QCD theory with $N \gg 1$ colors [10]. The magnitude of each of the $n!$ Feynman diagram making up A_n is of order N^n , since each boson can be emitted from any of the N constituents. It will be shown that the irreducible amplitude I_n corresponds to emission of all n pions from the *same* constituent, hence I_n is of order $N \equiv \mu$, independent of n . Unless there

are selection rules forbidding it, the individual bosons would like to be emitted from as many different constituents as possible to maximize the matrix element. This leads to the classical approximation used in the McLerran-Venugopalan model [8]. In the case when A_n describes a scattering amplitude with one incoming and $n - 1$ outgoing pions, terms in the decomposition with $k > 1$ vanish. This is so because at least one of the k irreducible factors must contain only outgoing pions, and by energy conservation this cannot survive. Since the only surviving term has $k = 1$, A_n is of order $\mu = N$ [11], rather than the magnitude N^n of each of its Feynman diagrams. This enormous cancelation even for a moderate n is not something one can easily obtain by calculating directly from the Feynman diagrams.

In Sec. 4 we consider parton diffractive scattering in perturbative QCD, in the limit of small coupling $\alpha_s \ll 1$, and high energy \sqrt{s} so that $\alpha_s \ln s = O(1)$. The scattering amplitude $\tilde{A}(s, b)$ (where b is the impact parameter) is no longer a tree, but it can still be decomposed into irreducible amplitudes $\delta_k(s, b)$ using the tree decomposition for each of the two fast partons. The irreducible amplitude δ_k is characterized by the exchange of k mutually interacting quasi-particles, and is of order α_s^k , so once again we have a factor $\mu = \alpha_s$ per quasi-particle. This observation will be used to devise a way to unitarize the BFKL-Pomeron amplitude [12], so that the total cross section when extrapolated to asymptotic energies obey the Froissart bound [13].

So far we have neglected the recoil of the energetic source. We will consider how to include it in the abelian situation in Sec. 5. This is prompted by the Landau-Pomeranchuk-Migdal (LPM) effect in QED [14], which describes a suppression of radiation for a particle traversing in a dense medium. More specifically, consider an energetic electron with energy $E = p_1^0$ moving in a medium of scatterers spaced apart by a distance d . Let the energy of the emitted photon be $\omega = k^0 = xE$, and the energy of the final electron be $p_2^0 = (1 - x)E$. Neglecting the mass of the charged particle, the longitudinal momentum transfer in such a process is $q^3 = p_2^3 + k^3 - p_1^3 \simeq (p_{2\perp}^2/(1 - x) + k_{\perp}^2/x)/(2E)$. If $|q^3 d| \ll 1$, emission from different scatters is coherent, as if it came from one source instead of many. This causes a suppression of radiation compared to the case when the emission is incoherent, when the

intensity of emission is proportional to the number of scatters. This suppression is the LPM effect. Since $q^3 = O(1/E)$, we must also compute the influence of scatters to $O(1/E)$ for consistency, and as we shall see in Sec. 5, this calls for a recoil correction to the same order.

II. RECOILLESS EMISSION

A. Propagator

Consider a relativistic source moving parallel to the z-axis at nearly the speed of light. Its transverse position x_\perp and its lightcone distance $x^- \equiv x^0 - x^3$ are fixed provided the emission of particles causes no recoil. This classical picture holds even quantum mechanically for the following reason. Let p'^α be the on-shell momentum of the energetic source after all emissions ($p \cdot p = m^2$), and k_i^α the outgoing momentum of the i th boson (Fig. 1), with $|k_i^\alpha| \ll p'^0$ for every α and every i . Then the propagators of Fig. 1 can be approximated by

$$P(p' + K) \equiv \frac{1}{(p' + K)^2 - m^2 + i\epsilon} \simeq \frac{1}{2p' \cdot K + i\epsilon}, \quad (1)$$

where K is an appropriate sum of k_i 's and the term $K \cdot K$ has been neglected compared to $2p' \cdot K$. The propagator in configuration space is obtained by Fourier transform to be

$$\begin{aligned} \tilde{P}(x) &\equiv \frac{1}{(2\pi)^4} \int d^4(p' + K) e^{-i(p' + K) \cdot x} \frac{1}{(p' + K)^2 - m^2 + i\epsilon} \\ &\simeq -\frac{i}{p'^+} e^{-ip' \cdot x} \delta^2(x_\perp) \delta(x^-) \theta(x^+), \end{aligned} \quad (2)$$

where the lightcone coordinates are defined by $A^\pm = A^0 \pm A^3$. With this definition and with the mass of the source neglected, the only non-zero component of the source momentum may be taken to be $p'^+ \equiv 2E$. When the source particle propagates from x_1 to x_2 with $x = x_1 - x_2$, this expression confirms that the transverse and the lightcone positions of the particle are fixed, while its lightcone time x^+ goes forward as it propagates, as in the classical picture.

Note that if the Feynman propagator in (1) is replaced by the Cutkosky propagator $-2\pi i \delta[(p + K)^2 - m^2]$, then the factor $\theta(x^+)$ in (2) is replaced by 1.

For a non-relativistic source with $|K^\mu| \ll m$, the corresponding propagator becomes

$$-\frac{i}{2m}e^{-imx^0}\delta^3(\vec{x})\theta(x^0). \quad (3)$$

B. Decomposition and Factorization

To conform to Bose-Einstein statistics, the emitted bosons in Fig. 1 must be symmetrized, with the full n -boson amplitude A_n given by a sum of $n!$ permuted diagrams. Labeling these permuted diagrams by the order of their bosonic emissions, as in $[\sigma] = [\sigma_1\sigma_2\cdots\sigma_n]$, so that Fig. 1 is $[12\cdots n]$, we have

$$A_n = \sum_{[\sigma] \in S_n} A[\sigma] = \sum_{[\sigma] \in S_n} V[\sigma]P[\sigma], \quad (4)$$

where the sum is taken over the set S_n of all permutations of n objects. Each $A[\sigma]$ is given by the product of the vertex factor $V[\sigma] = V_{\sigma_1}V_{\sigma_2}\cdots V_{\sigma_n}$, and the product $P[\sigma]$ of all the propagators. If the initial source particle is off-shell, there are n propagators like (1). If it is on-shell, then there are only $n - 1$ propagators, but it is convenient to include also an explicit on-shell δ -function factor $-2\pi i\delta(p' \cdot p' - m^2)$ into A_n , where $p' = p + \sum_{i=1}^n k_i$ is the momentum of the initial source particle. In the no-recoil approximation, the argument of this Cutkosky propagator becomes $p' \cdot p' - m^2 \simeq 2p \cdot \sum_{i=1}^n k_i$.

In configuration space, each Feynman propagator is given by (2). The Cutkosky propagator is given by the same formula with $\theta(x^+)$ replaced by 1. We shall denote the product of propagators in the configuration space by $\tilde{P}[\sigma]$, so that the configuration-space amplitude is

$$\tilde{A}_n = \sum_{[\sigma] \in S_n} V[\sigma]\tilde{P}[\sigma]. \quad (5)$$

To establish the decomposition formula, the most important factor in $\tilde{P}[\sigma]$ is $\theta[\sigma] = \prod_{i=1}^n \theta(x_i^+ - x_{i+1}^+)$, where $x_{n+1} \equiv x_0$ is the spacetime coordinate of the initial source if it is off-shell, and it is $-\infty$ if it is on-shell. We shall denote the amplitude by \tilde{A}'_n when $\tilde{P}[\sigma]$ is replaced by $\theta[\sigma]$.

Before we proceed note that the propagator in (1) contains no numerator factor. This implies that either helicity is conserved as in a vector-boson emission from a relativistic source, or that the spin-flip information is encoded in the vertex factor V_i as in non-relativistic Yukawa interaction. To illustrate how the numerator can formally be made to disappear, suppose the energetic source to be a spin- $\frac{1}{2}$ particle. If $p + K$ is the momentum of the propagator, as in (1), then the numerator is $\gamma \cdot (p + K) + m \simeq \gamma \cdot p = \sum_{\lambda} u_{\lambda}(p) \bar{u}_{\lambda}(p)$. We then move the u factor leftward and the \bar{u} factor rightward, and absorb them into the vertex factors V_{σ_i} . In this way the numerator factor can be made to disappear.

To explain the decomposition formula, look at the simplest case when $n = 2$:

$$\begin{aligned} \tilde{A}'_2 &= \theta[12]V_1V_2 + \theta[21]V_2V_1 \\ &= (\theta[12] + \theta[21])V_1V_2 + \theta[21][V_2, V_1] \\ &= (\theta[1]V_1)(\theta[2]V_2) + \theta[21][V_2, V_1], \end{aligned} \tag{6}$$

where the identity $\theta[12] + \theta[21] = \theta[1]\theta[2] \equiv \theta(x_1^+ - x_0^+)\theta(x_2^+ - x_0^+)$ has been used. We shall now write this formula in a way that can be generalized to all n .

For that purpose introduce the *quasi-particle amplitude*

$$\tilde{Q}'[\sigma] = \theta[\sigma][V_{\sigma_1}, [V_{\sigma_2}, [\cdots, [V_{\sigma_{n-1}}, V_{\sigma_n}] \cdots]]], \tag{7}$$

which is obtained from the Feynman amplitude $\tilde{A}'[\sigma]$ by replacing the products in $V[\sigma]$ by their nested commutators. If $\sigma', \sigma'', \cdots, \sigma'''$ are sequences of non-overlapping numbers, we will also let

$$\tilde{Q}'[\sigma'|\sigma''|\cdots|\sigma'''] = \tilde{Q}'[\sigma']\tilde{Q}'[\sigma'']\cdots\tilde{Q}'[\sigma''']. \tag{8}$$

In other words, vertical bars (or *cuts*) are used to separate the factors into products.

With this notation, we can rewrite (6) to be

$$\tilde{A}'_2 = \tilde{Q}'[1|2] + \tilde{Q}'[21]. \tag{9}$$

The general decomposition formula is given by [1]

$$\tilde{A}'_n = \sum_{[\sigma] \in S_n} \tilde{Q}'_c[\sigma], \quad (10)$$

where $\tilde{Q}'_c[\sigma]$ is by definition equal to $\tilde{Q}'[\sigma]$ with vertical bars suitably inserted into the argument $[\sigma]$. The rule for insertion is the following: a vertical bar is inserted after the number σ_i in $[\sigma]$ if and only if there is no number to its right smaller than it. Thus for example $\tilde{Q}'_c[12] = \tilde{Q}'[1|2]$, and $\tilde{Q}'_c[21] = \tilde{Q}'[21]$, so (9) is a special case of (10). More complicated examples for $n = 8$ are: $\tilde{Q}'_c[64312857] = \tilde{Q}'[6431|2|85|7] = Q'[6431]Q'[2]Q'[85]Q'[7]$, and $Q'_c[12385476] = Q'[1|2|3|854|76] = Q'[1]Q'[2]Q'[3]Q'[854]Q'[76]$. These are illustrated graphically in Fig. 2, where a cut denotes factorization in the configuration space. We have also shown simplified versions where each quasi-particle is indicated by a single thick vertical line. Note however from the remark in the paragraph following (2) that in momentum space, a vertical means replacing the Feynman propagator by the corresponding Cutkosky propagator.

To obtain the decomposition formula for the full configuration-space amplitude \tilde{A}_n , we simply remove the prime on both sides of (10). To get the decomposition formula for the momentum-space amplitude A_n , we will remove the tilde on both sides as well. With or without tilde and/or prime, Q_c is always related to Q by the same rule of insertion of vertical bars. It is the multi-quasi-particle amplitude (also known as the *non-abelian cut amplitude*) that factorizes according to the vertical bars into products of the irreducible single-quasi-particle amplitudes $Q[\sigma]$. The irreducible amplitude $Q[\sigma]$ (without vertical bars) is always obtained from the Feynman amplitude $A[\sigma]$ by replacing the products of V_i 's with their nested commutators. These are the quantities I_m mentioned in the Introduction, with m being the length of $[\sigma]$.

It is easy to incorporate the decomposition formula into Feynman diagrams by a slight modification of the latter. All that one has to do is to add cuts to the energetic tree(s) according to the insertion rules. Figs. 2 and 4 are examples of such *non-abelian cut diagrams* [15]. Ordinary Feynman rules remain unchanged, except for two modifications. A cut propagator is a Cutkosky propagator instead of a Feynman propagator, and the product

of vertex factors V_i between cuts are changed into their nested commutators. Note that individual Feynman diagrams are not the same as individual cut diagrams, but according to (4) and (10) their permuted sums are equal.

III. EMISSION FROM A COMPOSITE SOURCE

Consider a non-relativistic source made up of N constituents, with ψ^\dagger, ψ being the creation and annihilation operators for these constituents. Bosons interact with the source through a vertex of the form $V_i = \lambda \psi^\dagger \Gamma_i \psi$. The matrix element of V_i is of order λN , since each boson can be emitted from any of the N constituents of the source. See Fig. 3. If we have a product of m such operators, $V_1 V_2 \cdots V_m$, then its matrix element is of order $(\lambda N)^m$. In contrast, since $[V_i, V_j] = \psi^\dagger [\Gamma_i, \Gamma_j] \psi$, the matrix element of nested commutators of V_i is of order $\lambda^m N$. We may therefore identify the common magnitude μ with this number N , if $\lambda = O(1)$. Since this nested commutator is still given by a one-body operator, we may interpret the irreducible or quasi-particle amplitude I_m as describing emissions of the m bosons from the same constituent in the source.

As discussed in the Introduction, we may now conclude that the amplitude A_n for the inelastic reaction

$$\pi_1 + S \rightarrow \pi_2 + \pi_3 + \cdots + \pi_n + S, \quad (11)$$

where the non-relativistic source S is either a heavy nucleus or a baryon with $N \gg 1$ colors, is of order $\lambda^n N$, instead of $(\lambda N)^n$ when the reaction is described by tree amplitudes like Figs. 1 and 3.

In particular, for $n = 1$, we obtain the effective coupling of the pion to the source to be $g = \lambda N$. If the source S is a baryon with a large color N , it is known [10] that $\lambda \sim 1/\sqrt{N}$ so the Yukawa coupling constant is $g \sim \sqrt{N}$. The interaction is then very strong. Under such a circumstance, each Feynman tree diagram is of order $(\lambda N)^n \sim N^{n/2} \gg 1$, and loop diagrams will be even larger. There seems to be no reason at all to be able to calculate the

process using tree diagrams alone, yet doing so fairly realistic baryons can be obtained [16]. What happens is that although individual tree diagrams are large, their sum gives rise to a small amplitude $A_n \sim \lambda^n N \sim N^{1-n/2}$, so loop diagrams computed from A_n are negligible. Moreover, for large N , meson-meson direct couplings are small, so sums of tree diagrams like Figs. 1 and 3 indeed dominates the whole reaction process in (11) [11].

IV. PARTON-PARTON DIFFRACTIVE SCATTERING

In order to appreciate what decomposition can do for this process, let us first summarize what is known about the QCD calculation of this amplitude, at high energy \sqrt{s} and small momentum-transfer $\Delta = \sqrt{-t}$.

Even assuming the QCD coupling α_s to be very small, each loop integration is capable of producing a factor $\ln s$, so the additional factor for each loop is likely to be $\alpha_s \ln s$ and not just α_s . If $\alpha_s \ln s = O(1)$ which we shall assume from now on, loops of every order must be computed, and then summed. Such a difficult task can usually be attempted only in the leading-log approximation, where the highest power of $\ln s$ is kept at each order. This is equivalent to keeping only the lowest power of α_s for fixed $\alpha_s \ln s$.

Such leading-log calculations have been carried out [12]. The dominant contribution is of order α_s (for fixed $\alpha_s \ln s = O(1)$), and is mediated by the exchange of a color-octet object known as the reggeized gluon, or Reggeon for short. For a truly elastic scattering one needs a color-singlet exchange. That amplitude is of order α_s^2 and the effective object being exchanged is known as a BFKL Pomeron. The dependence on $\alpha_s \ln s$ is also known. This dependence leads to a total cross section $\sigma_{tot}(s)$ growing like $s^{12(\ln 2)\alpha_s/\pi}$. Extrapolated to a large s , such a power growth violates unitarity and the Froissart bound, which forbids any total cross section to grow faster than $\ln^2 s$.

This problem arises because we extrapolate the BFKL Pomeron amplitude, correct for $\alpha_s \ll 1$ and $\alpha_s \ln s = O(1)$, beyond its region of validity to very large s . To cure the problem we must add the necessary subdominant terms before extrapolation. This is not unlike

extrapolating the first Born approximation valid for weak coupling, to a strong-coupling regime where contributions from higher Born terms must be included. At high energy this task is somewhat simplified because there is an impact-parameter representation for the amplitude

$$A(s, \Delta) = 2is \int d^2b e^{i\Delta \cdot \mathbf{b}} \tilde{A}(s, b),$$

$$\tilde{A}(s, b) = 1 - e^{2i\delta(s, b)}. \quad (12)$$

Born approximation corresponds to small phase shift $\delta(s, b)$, whence the Born amplitude $\tilde{A}(s, b)$ in the impact-parameter space is proportional to $\delta(s, b)$. Higher Born approximations are given simply by powers of the phase shift. To simulate the problem encountered by the BFKL Pomeron, consider an interaction mediated by the exchange of a spin- $J > 1$ particle with an interaction range μ^{-1} . The phase shift at large b has the form $\delta(s, b) = c(s/s_0)^{J-1} \exp(-\mu b)$. It is small if $c \ll 1$ and $s \sim s_0$. When s becomes large, the phase shift is no longer small, so the full expression (12) has to be used. The Born approximation violates the Froissart bound when extrapolated to large s , but the full expression can be shown not to [13].

This would be a good way to cure the problem of the BFKL Pomeron if it could be interpreted as a phase shift. For this scenario to be true it is necessary to demonstrate that the correction terms are given by powers of the phase shift. Unfortunately this is difficult to do, for factorization is not easy to prove by the usual means, and in any case in the region $\alpha_s \ll 1$ and $\alpha_s \ln s = O(1)$ where we have control, the correction terms are small and difficult to calculate. However, as discussed in the previous sections, multi-quasi-particle amplitudes do factorize, and within that formalism subdominant terms can also be calculated just by multiplying a number of dominating terms. So with the help of the decomposition formula it is at least hopeful that we may be able to interpret the BFKL-Pomeron amplitude as a phase shift.

The reality is actually more complicated. We shall summarize here what transpires and the detail can be found in Ref. [13]. First we must find a decomposition formula

suitable for the parton-parton amplitude, by applying the result of Sec. 2 to each of the two energetic partons. In the region of interest, $\alpha_s \ll 1$ and $\alpha_s \ln s = O(1)$, it turns out that the amplitude $\tilde{A}(s, b)$ can be decomposed into sums of products of irreducible amplitudes $2i\delta_k(s, b)$. This is analogous to the situation in Sec. 2 where the tree amplitude $\sum_n A_n$ can be decomposed into sums of products of the irreducible amplitudes $Q[\sigma]$. The difference is that each $Q[\sigma]$ is a single-quasi-particle amplitude, but $\delta_k(s, b)$ is characterized by k mutually interacting quasi-particles being exchanged in the t -channel, as illustrated in Fig. 4. The difference arises because gluon interaction was ignored in the tree amplitude considered in Sec. 2. When included, gluons within the same quasi-particle as well as those in different quasi-particles may interact, which is why single-quasi-particle amplitudes do not necessarily factorize anymore. Nevertheless, each quasi-particle is still a color octet, and each would be associated with a common magnitude of the order of $\mu = \alpha_s$, in the sense that

$$\delta_k(s, b) = O(\alpha_s^k). \quad (13)$$

The factorized multi-quasi-particle amplitudes of all orders can be summed up to an exponential form to yield the impact-parameter representation (12), with the phase shift given by

$$\delta(s, b) = \sum_{k=1}^{\infty} \delta_k(s, b). \quad (14)$$

To connect this with the BFKL-Pomeron amplitude, let us first use (13) to conclude that the dominant amplitude for $\tilde{A}(s, b)$, in the region $\alpha_s \ll 1$ and $\alpha_s \ln s = O(1)$, comes from δ_1 and is of order α_s . In this regime, $\tilde{A}(s, b) \simeq -2i\delta_1(s, b)$ is mediated by the exchange of a single color-octet quasi-particle. As stated in the beginning of this section, such a dominant octet amplitude is the Reggeon amplitude, so we can simply identify the totality of single quasi-particles with a Reggeon. To obtain a color-singlet amplitude, at least two quasi-particles must be exchanged, in which case $\tilde{A}(s, b)$ is dominated by a combination of δ_1^2 and δ_2 , both of order α_s^2 . The first term describes two non-interacting quasi-particles, and the second two mutually interacting ones. We may therefore identify the BFKL-Pomeron amplitude with

the color-singlet component of this two-quasi-particle amplitude. Thus Feynman diagrams can be summed up to give a unitary formula (12) respecting the Froissart bound, and the phase shifts δ_1 and δ_2 can be obtained from the leading-log Reggeion and the BFKL-Pomeron amplitudes. We might want to drop the δ_k for $k > 3$ on the grounds that they are subdominant according to (13), but they may no longer be small when s is extrapolated.

V. RECOIL CORRECTION

So far the propagator has been taken to be (2) where all recoils are ignored. Motivated by the need of the Landau-Pomeranchuk-Migdal (LPM) effect [14], we shall study in this section how to include recoil. As mentioned in the Introduction, the LPM effect is of order $1/p^+$, so recoil correction to this order has to be taken into account. In what follows we shall discuss how recoil can be incorporated into the propagator, how this correction affects abelian factorization, and finally how the new factorization can be summed up into an exponential form for the wave function. We will not carry this forward to discuss the LPM effect as such because discussions can be found elsewhere [14,17]. The factorization and the subsequent rendering of an exponential form is important because phase information, which is crucial for the radiation suppression contained in the LPM effect, is contained in the exponent. The treatment below parallels the discussion in Ref. [17], but it is presented here in the configuration space, making it both algebraically simpler and intuitively more evident.

A. Propagator

Let us return to the inverse propagator in its exact form, $(p + K)^2 - m^2 + i\epsilon = (p + K)^+ K^- - k_\perp^2 + i\epsilon$. We proceed to change $(p + K)^+$ to p^+ , on the grounds that $p^+ \gg K^+$, but make no further approximations at this point. In that case instead of (1) we have $[p^+ K^- - k_\perp^2 + i\epsilon]^{-1}$, whose Fourier transform gives the propagator in configuration space to be

$$\begin{aligned}
\tilde{P}(x) &= \frac{1}{(2\pi)^4} \int d^4(p+K) e^{-i(p+K)\cdot x} \frac{1}{(p+K)^2 - m^2 + i\epsilon} \\
&\simeq -\frac{i}{(2\pi)^2 p^+} \int d^2 k_\perp e^{-ip\cdot x - ik_\perp^2 (x^+/p^+) - ik_\perp \cdot x_\perp} \delta(x^-) \theta(x^+) \\
&= -\frac{1}{4\pi x^+} e^{-ip\cdot x + i(p^+/4x^+)x_\perp^2} \delta(x^-) \theta(x^+).
\end{aligned} \tag{15}$$

The Gaussian function in x_\perp has a rms width proportional to x^+/p^+ , so we may think of it as a result of random walk growing with the lightcone time x^+ . To the first subleading order, we may approximate it by

$$-\frac{1}{4\pi x^+} \exp \left[i(p^+/4x^+)x_\perp^2 \right] \simeq -\frac{i}{p^+} \left(\delta^2(x_\perp) + \frac{x^+ i}{p^+} \nabla^2 \delta^2(x_\perp) \right), \tag{16}$$

where ∇^2 is the two-dimensional Laplacian in the transverse variables. The first term in (16) is the recoilless contribution given in (2). The second term gives the recoil correction needed for the LPM effect.

B. Leading-Order Wave Function

Consider a charged particle with relativistic momentum p moving in a background vector potential $A^\mu(x)$. The i th scattering vertex is given by $V_i = 2p \cdot A(x_i) = p^+ A^-(x_i)$. In this subsection we will review how to compute the wave function of the charged particle in the leading order. In the next section we will study how the first subleading corrections can be included.

The outgoing wave function $\psi(x_0)$ can be obtained from the configuration-space off-shell amplitude of (5) by

$$\psi(x_0) = e^{-ip\cdot x_0} \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^n d^3 x_i \tilde{A}_n, \tag{17}$$

where $n!$ is the symmetry factor for identical bosons. Since the vertex is abelian, the result of the permutation summation in (5) is just to replace the $\theta(x^+)$ factors of (2) in all the

propagators by $\prod_{i=1}^n \theta(x_i^+ - x_0^+)$. For simplicity we shall drop the subscript 0 in the coordinates of the wave function, and write $\frac{1}{2}A^-(x)$ as $V(x)$. Then the outgoing wave function in the leading order is

$$\begin{aligned}\psi^{(0)}(x) &= e^{-ip \cdot x} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \chi_0(x)^n \\ &= \exp(-ip \cdot x - i\chi_0(x)),\end{aligned}\tag{18}$$

where

$$\chi_0(x) = \frac{1}{2} \int_{x^+}^{\infty} dx^{+'} A^-(x^{+'} x^- x_{\perp}) \equiv \int_{x^+}^{\infty} dx^{+'} V(x')\tag{19}$$

describes the phase shift it accumulates as the particle moves down its path. The integration variable x' is understood to have the same minus and perpendicular components as x . The same convention will be used for the integration variables x' and x'' later in equation (22).

C. Subleading Correction

When the subleading term of propagator (16) is taken into account, both factorization and exponentiation become more involved, owing to the presence of the Laplacian operator in $\nabla^2 \delta^2(x_{i\perp} - x_{i+1\perp})$, as well as the additional factor $(x_i^+ - x_{i+1}^+)$ in the correction term. The saving grace is that we need to compute the correction only to the first subleading order $1/p^+$, so these corrections come in only linearly. Then the correction term is

$$\psi^{(1)}(x) = \frac{i}{p^+} e^{-ip \cdot x} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{x^+}^{\infty} \prod_{k=1}^n dx_k^+ \mathcal{C}_n,\tag{20}$$

where

$$\begin{aligned}\mathcal{C}_n &= \sum_{i=1}^n \left[\nabla^2 \left(\prod_{k=1}^{i-1} \theta(x_k^+ - x_i^+) V(x_k^+) \right) \right. \\ &\quad \left. \theta(x_i^+ - x_{i+1}^+) (x_i^+ - x_{i+1}^+) \prod_{l=i}^n V(x_l^+) \right] \\ &= \sum_{i=1}^n (x_i^+ - x^+) \nabla^2 V(x_i^+) \prod_{j \neq i} V(x_j^+) + \\ &\quad 2 \sum_{i < j} \nabla V(x_i^+) \cdot \nabla V(x_j^+) \theta(x_i^+ - x_j^+) (x_j^+ - x^+) \prod_{l \neq i, j} V(x_l^+).\end{aligned}\tag{21}$$

Thus

$$\begin{aligned}
\int_{x^+}^{\infty} \prod_{k=1}^n dx_k^+ \mathcal{C}_n &= \chi_0^{n-2}(x) [n\chi_0(x)\chi_2(x) + n(n-1)\chi_1(x)], \\
\chi_2(x) &\equiv \int_{x^+}^{\infty} dx^{+'} (x^{+'} - x^+) \nabla^2 V(x^{+'}) \\
&= \nabla^2 \int_{x^+}^{\infty} dx^{+'} \chi_0(x'), \\
\chi_1(x) &\equiv \int_{x^+}^{\infty} dx^{+'} (x^{+'} - x^+) \nabla V(x') \cdot \int_{x_+}'^{\infty} dx^{+''} \nabla V(x'') \\
&= \int_{x^+}^{\infty} dx^{+'} [\nabla \chi_0(x')]^2.
\end{aligned} \tag{22}$$

Summing up n , we find to the accuracy of order $1/p^+$ that the outgoing wave function is given by

$$\begin{aligned}
\psi(x) &= \psi^{(0)}(x) + \psi^{(1)}(x) \\
&= \exp \left[-ip \cdot x - i\chi_0(x) - \frac{i}{p^+} (\chi_1(x) + i\chi_2(x)) \right].
\end{aligned} \tag{23}$$

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FIGURES

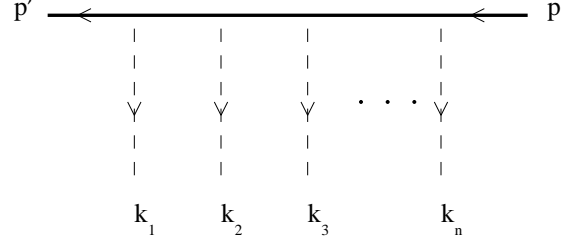


FIG. 1. A tree diagram showing bosons of momenta k_i being emitted from an energetic source with initial momentum p and final momentum p' .

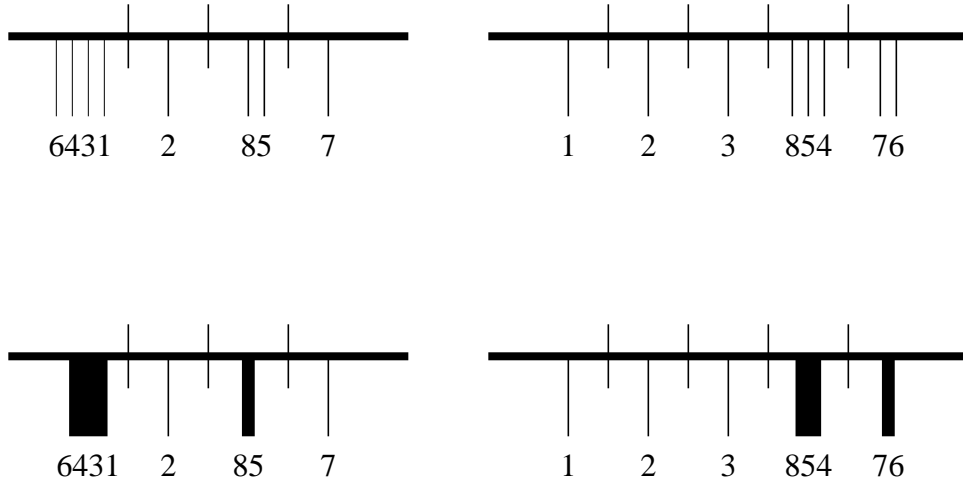


FIG. 2. Two $n = 8$ non-abelian cut diagrams showing where cuts should appear. For each of them, a simplified version is given in which every quasi-particle is shown as a thick vertical line.

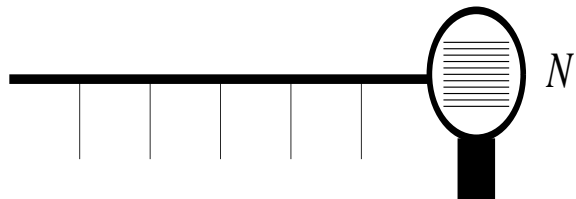


FIG. 3. A tree diagram whose source has N constituents.

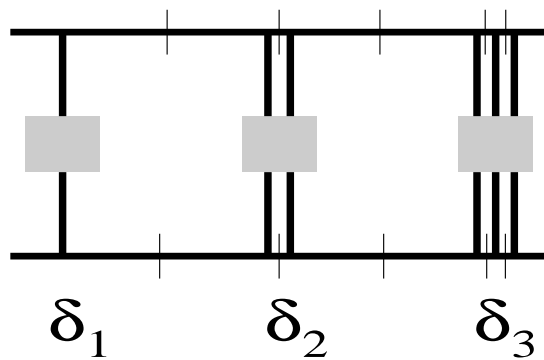


FIG. 4. The decomposition of a parton-parton scattering amplitude into products of irreducible amplitudes δ_k , where k indicated the number of mutually interacting quasi-particles (thick vertical line) being exchanged.